CRANBROOK SCHOOL

Student Number: ____________________

2023 **HSC TRIALS - ASSESSMENT TASK 4**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 120 minutes
- Write using blue or black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total Marks: 70

Section I – 10 marks (pages 2–5)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 6–9)

- Attempt Questions 11–14
- Allow about 105 minutes for this section

Section I: Multiple Choice

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

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1 The polynomials $P(x)$ and $Q(x)$ are such that $P(x) = (x+3)(x-2)Q(x)$. When $P(x)$ is divided by $(x+1)$, the remainder is 12. What is the remainder when $Q(x)$ is divided by $(x+1)$?

A. -3

B. -2

C. 2

D. 3

2 Which of the following is equivalent to $\dfrac{d}{dx}\left(2\sin^{-1}\dfrac{x}{2}\right)$?

A. $\dfrac{1}{\sqrt{1-x^2}}$

B. $\dfrac{2}{\sqrt{1-x^2}}$

C. $\dfrac{2}{\sqrt{4-x^2}}$

D. $\dfrac{1}{2\sqrt{4-x^2}}$

3 What is the coefficient of x^5 to $(3+2x)^7$

A. 21

B. 128

C. 1344

D. 6048

4 What is the maximum value of $15\sin\theta - 8\cos\theta$?

A. 7

B. 15

C. 17

D. 23

5 What is the range of the function $f(x) = \tan^{-1}(\sin x)$?

A. $\left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$

B. $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$

C. $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

D. $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

6 Which of the following gives the complete domain of $y = 5\cos^{-1}\left(\frac{2-x}{3}\right)$?

A. $[1, 5]$

B. $[-1, 5]$

C. $[-5, 1]$

D. $[-5, -1]$

7 Consider the function $f(x) = x^2 + 2$, with domain $x \geq 0$.
What is the gradient of the tangent to $y = f^{-1}(x)$ at $x = 3$

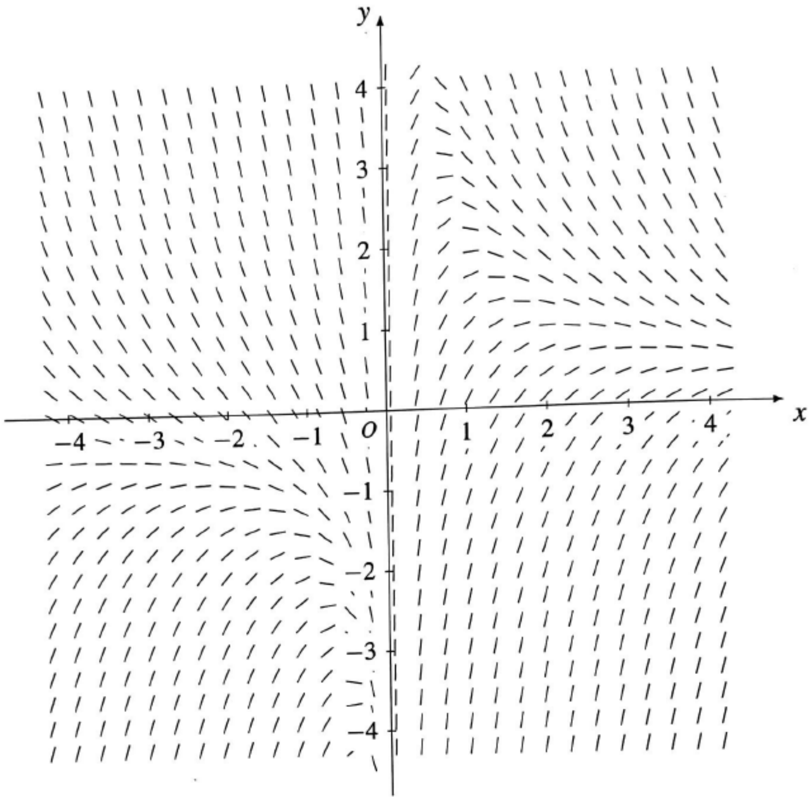
A. $\frac{1}{6}$

B. $\frac{1}{2}$

C. 2

D. 6

8 The slope field for a differential equation is sketched below.



Which of the following is the correct differential equation for the slope field?

A. $\dfrac{dy}{dx} = \dfrac{2}{x} - y$

B. $\dfrac{dy}{dx} = \dfrac{2}{x} + y$

C. $\dfrac{dy}{dx} = \dfrac{2}{y} - x$

D. $\dfrac{dy}{dx} = \dfrac{2}{y} + x$

9 A school principal calls a meeting with two student representatives from each year group from Year 7 to Year 12. The 12 students and the principal sit at a round table. In how many ways can the 13 participants sit around the table if at least one pair of students from a year group sits apart?

A. $12! - 6! \times 2^6$

B. $13! - 6! \times 2^6$

C. $12! - 5! \times 2^5$

D. $13! - 5! \times 2^5$

10 Let $f(x)$ be a continuous function with $a > 0$ and $k > 0$. Which of the following is true?

A. $\displaystyle\int_0^a f(x)\,dx = k\int_0^{ak} f(kx)\,dx$

B. $\displaystyle\int_0^a f(x)\,dx = \frac{1}{k}\int_0^{ak} f(kx)\,dx$

C. $\displaystyle\int_0^a f(x)\,dx = k\int_0^{ak} f(\frac{x}{k})\,dx$

D. $\displaystyle\int_0^a f(x)\,dx = \frac{1}{k}\int_0^{ak} f(\frac{x}{k})\,dx$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour 45 minutes for this section

Answer each question in this section in the separate writing booklet provided.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks)

(a) Solve: $|5x-1|<5$ **2**

(b) $\displaystyle\int \frac{1}{9+25x^2}dx$ **2**

(c) Find The polynomial $P(x)=x^3+3x^2-4x-5$ has roots α,β and γ

(i) Find $\alpha+\beta+\gamma$ **1**

(ii) Find $\alpha^2+\beta^2+\gamma^2$ **2**

(d) For the vectors $\underset{\sim}{u}=2\underset{\sim}{i}+3\underset{\sim}{j}$, and $\underset{\sim}{v}=5\underset{\sim}{i}-12\underset{\sim}{j}$, find each of the folllowing:

(i) $\underset{\sim}{u}\cdot\underset{\sim}{v}$ **2**

(ii) $|\underset{\sim}{v}|$ **1**

(iii) the vector projection of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$ **2**

(e) In how many ways can the letters of the word POSSIBILITY be arranged so that the two Ss are together. **2**

Question 12 (15 marks)

(a) Solve the differential equation $\frac{dy}{dx} = \frac{2}{x^3 e^y}$, where $y(1) = 0$. **3**

Express your solution in the form $y = f(x)$

(b) Prove the identity $\tan\left(\frac{\pi}{4} + x\right) - \tan\left(\frac{\pi}{4} - x\right) = 2\tan(2x)$ **3**

(c) Given that $|\underset{\sim}{a}| = 2$, $|\underset{\sim}{b}| = 3$ and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 5$, calculate the length of $3\underset{\sim}{a} - 2\underset{\sim}{b}$. **3**

(d) Find the value of $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin x - \cos x)^2 \, dx$ **3**

(e) Use Mathematical induction to prove that $23^n - 1$ is divisible by 11, for all positive integers n. **3**

Question 13 (16 marks)

(a) (i) Show that $\dfrac{d}{dx}\tan^3 x = 3\sec^4 x - 3\sec^2 x$. **1**

(ii) Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sec^4 x\,dx$ **3**

(b) Find in the form $y = f(x)$ the solution of the differential equation **3**

$$\left(\frac{1}{y} + \frac{1}{1-y}\right)\frac{dy}{dx} = 1$$

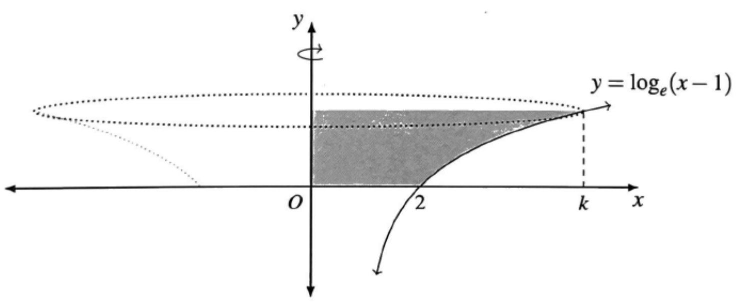
given $y = \dfrac{1}{2}$ when $x = 0$.

(c) (i) Show that $\cos x + \cos 5x = 2\cos 3x \cos 2x$ **1**

(ii) Hence or otherwise, solve $\cos x + \cos 5x = \cos 2x$ for $x \in [0, \pi]$. **2**

(d) Use the substitution $u = \tan^{-1} x$ to find $\displaystyle\int_0^{\sqrt{3}} \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$ **3**

(e) The shaded region below is bounded by the positive $x-$axes and $y-$axes, the curve $y = \log_e(x-1)$, and a horizontal line which intersects the curve where the $x-$coordinate is the rational number k. **3**



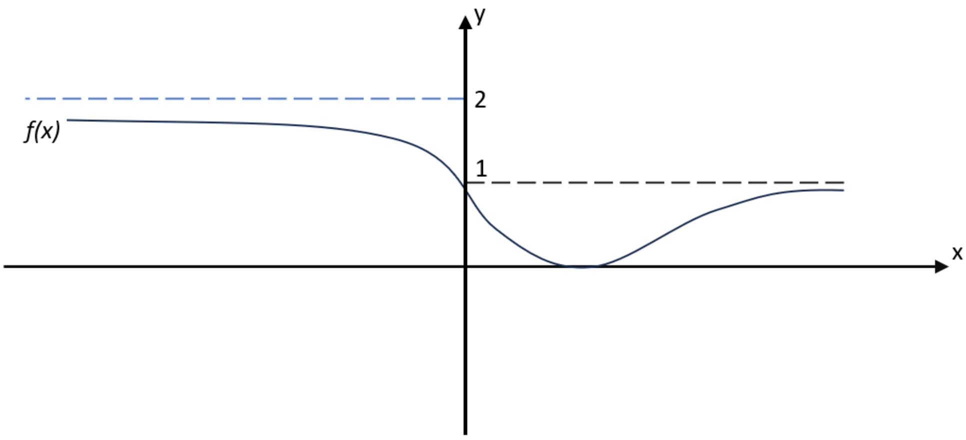
The volume V formed when this region is rotated around the $y-$axis is given by

$$V = \frac{\pi(27 + \log_e 16)}{2}$$

Find the value of k.

Question 14 (15 marks)

(a) The diagram below shows the graph of $y = f(x)$.



(i) Sketch $y = \sqrt{f(x)}$ **1**

(ii) Hence, sketch $y = \dfrac{1}{\sqrt{f(x)}}$ on a different diagram. **2**

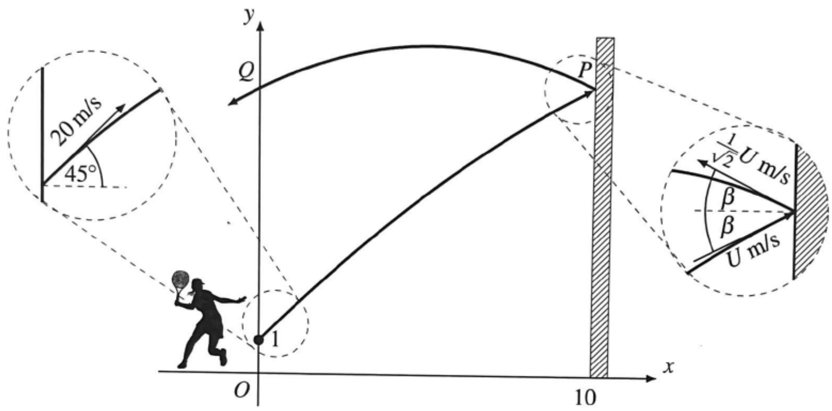
(b) A water tank is a prism with a base of area A square metres . Torricelli's Law states that the rate of change in the volume V cubic metres of a leaking tank with water level h metres at time t hours is given by the differential equation **5**

$$\frac{dV}{dt} = -h\sqrt{h}$$

.

If a tank has a base with area 10 square metres and the initial height of the water is 4 metres, find how long it takes until the tank contains less than 1 litre of water.

(c) Charlie hits a tennis ball at an angle of 45° against a vertical wall 10 m away. The ball rebounds off the wall and passes directly over where it was first struck at point Q. The tennis ball is 1m above the ground when Charlie hits it and has an initial speed of 20 m/s.



The position $\underset{\sim}{r}$ in metres of the ball t seconds after it is hit but before it hits the wall at P is given by the equation

$$\underset{\sim}{r}(t) = \begin{bmatrix} 10\sqrt{2}t \\ 10\sqrt{2}t - 5t^2 + 1 \end{bmatrix}$$

(i) Calculate the height at which the ball hits the wall. **1**

(ii) Find the velocity vector $\underset{\sim}{v}(t)$ of the ball during its flight from Charlie to the wall. **1**

(iii) Calculate the speed U m/s and the angle β at which the ball hits the wall. **2**

(iv) The ball rebounds off the wall at the same angle that it hits it, β but its speed is reduced so that it leaves the wall within an intial speed of $\dfrac{1}{\sqrt{2}}Um/s$. **3**
Show that the $y-$ coordinates of the points P and Q in the diagram above are equal. Assume the acceleration due to gravity is 10 m/s^2.

End of paper

RTAY Solns

Wednesday, 9 August 2023 10:36 am

11 a) $|5x-1|<5$

$-5 < 5x - 1 < 5$ (1)

$-4 < 5x < 6$

$-\frac{4}{5} < x < \frac{6}{5}$ (1) ✓

b) $\int \frac{1}{9+25x^2}\,dx$

$= \int \frac{1}{(3)^2+(5x)^2}\,dx$

Using $\int \frac{f'(x)}{a^2+[f(x)]^2} = \frac{1}{a}\tan^{-1}\left(\frac{f(x)}{a}\right)+C$

$= \frac{1}{5}\int \frac{5}{(3)^2+(5x)^2}\,dx$ (1)

$= \frac{1}{5}\left(\frac{1}{3}\tan^{-1}\left(\frac{5x}{3}\right)\right)+C$

$= \frac{1}{15}\tan^{-1}\left(\frac{5x}{3}\right)+C$ (1) ✓

c) i) $\alpha+\beta+\gamma = \frac{-b}{a}$

$= \frac{-3}{1}$

$= -3$ (1) ✓

ii) $(\alpha+\beta+\gamma)^2 = (\alpha+\beta+\gamma)(\alpha+\beta+\gamma)$

$= \alpha^2+\alpha\beta+\alpha\gamma+\beta\alpha+\beta^2+\beta\gamma+\gamma\alpha+\gamma\beta+\gamma^2$

$= \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\alpha\gamma)$ (1)

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= (-3)^2 - 2\left(\frac{-4}{1}\right)$

$= 9+8$

$= 17$ (1) ✓

d) i) $\underset{\sim}{u}\cdot\underset{\sim}{v} = x_1x_2+y_1y_2$

$= 2(5)+3(-12)$ (1)

$= 10-36$

$= -26$ (1) ✓

MC

1. B ✓
2. C ✓
3. D ✓
4. C ✓
5. B ✓
6. B ✓
7. B ✓
8. A ✓
9. A ✓
10. D ✓

$P(-1) = 12$

$(-1+3)(-1-2)Q(x) = 12$

$(2)(-3)Q(-1) = 12$

$-6Q(-1) = 12$

$Q(-1) = -2$

$\tan^{-1}(\sin x)$

$-1 \le \sin x \le 1$ $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

ii) $|\underset{\sim}{v}| = \sqrt{5^2 + (-12)^2}$

$= 13$ (1) ✓

iii) Proj of $\underset{\sim}{u}$ onto $\underset{\sim}{v}$ $\dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{v}|^2} \underset{\sim}{v}$

$= \dfrac{-26}{13^2} (5\underset{\sim}{i} - 12\underset{\sim}{j})$ (1)

$= -\frac{2}{13} (5\underset{\sim}{i} - 12\underset{\sim}{j})$ OR $-\frac{10}{13}\underset{\sim}{i} + \frac{24}{13}\underset{\sim}{j}$ (1)

e) P O S S I B I L I T Y

S : 2
I : 3
Letters : 11

$(11-1)!$ (1)

Bubble. Method.

No dividing by 2! as S's aren't distinguishable.

$\dfrac{10!}{3!}$ (1) ✓ $= 604800$

(12) a) $\dfrac{dy}{dx} = \dfrac{2}{x^3 e^y}$

$\int e^y \, dy = \int 2x^{-3} \, dx$ (1)

$e^y = \dfrac{2x^{-2}}{-2} + C$

$e^y = -\dfrac{1}{x^2} + C$ ✓

$y(1) = 0$

$e^0 = -\dfrac{1}{1^2} + C$

$1 + 1 = C$ (1)

does this need absolute values?

$e^y = -\frac{1}{x^2} + 2$ ✓

$y = \ln\left(-\frac{1}{x^2} + 2\right)$ (1)

b) $LHS = \tan\left(\frac{\pi}{4} + x\right) - \tan\left(\frac{\pi}{4} - x\right)$

$= \dfrac{\tan\left(\frac{\pi}{4}\right) + \tan x}{1 - \tan\left(\frac{\pi}{4}\right)\tan(x)} - \dfrac{\tan\left(\frac{\pi}{4}\right) - \tan(x)}{1 + \tan\left(\frac{\pi}{4}\right)\tan(x)}$

(1) expand double angle

(1) common denominator

(1) correct proof.

$= \dfrac{1 + \tan x}{1 - \tan x} - \dfrac{1 - \tan x}{1 + \tan x}$

$= \dfrac{(1 + \tan x)^2 - (1 - \tan x)^2}{1 - \tan^2 x}$

$1 + 2\tan x + \tan^2 x - (1 - 2\tan x + \tan^2 x)$

$\sin^2 x + \cos^2 x = 1$

$$\overline{\quad\quad\quad}$$
$$1 - \tan^2 x$$

$$= \frac{1 + 2\tan x + \tan^2 x - (1 - 2\tan x + \tan^2 x)}{1 - \tan^2 x}$$

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$$= \frac{4\tan x}{1 - \tan^2 x}$$

$$= \frac{4\tan x}{1 - \frac{\sin^2 x}{\cos^2 x}}$$

$$= \frac{\frac{4\sin x}{\cos x}}{\frac{\cos^2 x - \sin^2 x}{\cos^2 x}}$$

$$= \frac{4\sin x}{\cos x} \times \frac{\cos^2 x}{\cos^2 x - \sin^2 x}$$

$$= \frac{4\sin x\cos x}{\cos^2 x - \sin^2 x}$$

$$= \frac{2(2\sin x\cos x)}{\cos(2x)}$$

$$= \frac{2(\sin 2x)}{\cos(2x)}$$

$$= 2\tan(2x)$$

c) Distance $= |3\underset{\sim}{a} - 2\underset{\sim}{b}|$

Distance^2 $= |3\underset{\sim}{a} - 2\underset{\sim}{b}|^2$ (1) Progress towards expanding length of $3\underset{\sim}{a} - 2\underset{\sim}{b}$ or equiv merit.

$= (3\underset{\sim}{a} - 2\underset{\sim}{b}).(3\underset{\sim}{a} - 2\underset{\sim}{b})$ (1) Significant progress to finding length.

$= 9\underset{\sim}{a}^2 - 6\underset{\sim}{a}\underset{\sim}{b} - 6\underset{\sim}{a}\underset{\sim}{b} + 4\underset{\sim}{b}^2$ (1) Correct solution

$= 9|2|^2 - 6(5) - 6(5) \quad 4(3)^2$

$= 36 - 30 - 30 + 36$

$= 12$

$d = \sqrt{12}$

$d = 2\sqrt{3}$

d) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin x - \cos x)^2 \, dx$

$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^2 x - 2\sin x\cos x + \cos^2 x \, dx$ (1)

$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 1 - \sin 2x \, dx$

$= \left[x + \frac{1}{2}\cos 2x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$ (1)

$= \frac{3\pi}{4} + \frac{1}{2}\cos\left(\frac{3\pi}{4} \times 2\right) - \left(\frac{\pi}{4} + \frac{1}{2}\cos\left(2 \times \frac{\pi}{4}\right)\right)$

$= \frac{3\pi}{4} + \frac{1}{2}\cos\left(\frac{3\pi}{4} \times 2\right) - \left(\frac{\pi}{4} + \frac{1}{2}\cos\left(2 \times \frac{\pi}{4}\right)\right)$

$= \frac{\pi}{2} + \frac{1}{2}\cos\left(\frac{3\pi}{2}\right) - \frac{1}{2}\cos\left(\frac{\pi}{2}\right)$

$= \frac{\pi}{2} + \frac{1}{2}(0) - \frac{1}{2}(0)$

$= \frac{\pi}{2}$ (1) ✓

e) Prove true $n=1$

LHS $23^1 - 1$

$= 22$

$= 2(11)$

$= 11M$

$=$ RHS. (1)

Assume true for $n=k$

$23^k - 1 = 11M$ (1)

Prove true for $n=k+1$

LHS $= 23^{k+1} - 1$

$= 23 \times 23^k - 1$

$= 23 \times 23^k - 1 - 22 + 22$

$= 23 \times 23^k - 23 + 22$

$= 23(23^k - 1) + 22$

$= 23(11M) + 22$

$= 11(23M) + 22$

$= 11(23M + 2)$ ✓ (1)

$= 11R$ where R is a integer

As proved by MI.

(3) a) i) $\frac{d}{dx}(\tan x)^3 = \frac{d}{dx}(\tan x) \times 3(\tan x)^2$

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$= 3\sec^2 x \tan^2 x$

$= 3\sec^2 x(\sec^2 x - 1)$ need all steps (1) ✓

$= 3\sec^4 x - 3\sec^2 x$

ii) $\int_0^{\frac{\pi}{4}} \sec^4 x \, dx$

(1) Some progress

(2) Significant progress connecting part i) to integral.

(3) Correct soln.

$\frac{d}{dx}(\tan^3 x) + 3\sec^2 x = 3\sec^4 x$

$\frac{1}{3}\left[\frac{d}{dx}(\tan^3 x)\right] + \sec^2 x = \sec^4 x$

$\int_0^{\frac{\pi}{4}} \left(\frac{1}{3}\frac{d}{dx}[\tan^3 x] + \sec^2 x\right) dx$

$= \frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{d}{dx}(\tan^3 x) + \int_0^{\frac{\pi}{4}} \sec^2 x$

$= \frac{1}{3}\left[\tan^3(x)\right]_0^{\frac{\pi}{4}} + \left[\tan x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{3}[1-0] + (1-0)$

$= \frac{1}{3}+1$

$= \frac{4}{3}$ ✓

b) $\left(\frac{1}{y}+\frac{1}{1-y}\right)\frac{dy}{dx} = 1$

$\int\left(\frac{1}{y}+\frac{1}{1-y}\right)dy = \int 1\,dx$ 1

$\ln|y| + -\int \frac{-1}{1-y}\,dy = x + c$

$\ln|y| - \ln|1-y| = x + c$

$\ln|\tfrac{1}{2}| - \ln|1-\tfrac{1}{2}| = 0 + c$

$\ln|\tfrac{1}{2}| - \ln|\tfrac{1}{2}| = c$

$c = 0$

$\therefore \ln|y| - \ln|1-y| = x$

$\ln\left|\frac{y}{1-y}\right| = x$ 1

$\frac{y}{1-y} = e^x$

$y = e^x(1-y)$

$y = e^x - e^x y$

$y + e^x y = e^x$

$y(1+e^x) = e^x$

$y = \frac{e^x}{1+e^x}$ 1

c) i) $LHS = \cos x + \cos 5x$

Using $\cos A\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

$2\cos A\cos B = \cos(A-B) + \cos(A+B)$ 1 need to use.

$\therefore A+B = 5$

$A-B = 1$

$A = 3x$

$B = 2x$

$\therefore \cos x + \cos 5x$

$= \cos(3x-2x) + \cos(3x+2x)$ ✓

$= 2\cos(3x)\cos(2x)$

ii) $\cos x + \cos 5x = \cos 2x$

$2\cos(3x)\cos(2x) = \cos(2x)$ 1 Solve 1 part of eqn.

$2\cos(3x)\cos(2x) - \cos(2x) = 0$ 2 Need all correct solution.

$\cos(2x)[2\cos(3x) - 1] = 0$

$\cos 2x = 0 \qquad 2\cos(3x) = 1$

$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \qquad \cos(3x) = \frac{1}{2}$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \qquad 3x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}$

$x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}.$

$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

$x = \frac{\pi}{4}, \frac{3\pi}{4},$

$3x = \frac{}{3}, \frac{}{3}, \frac{}{3}, 3, 3, 3$

$x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}.$

$\therefore x = \frac{}{4}, \frac{3\pi}{4}, \frac{}{9}, \frac{}{9}, \frac{}{9}$

d) i) $\int_0^{\sqrt{3}} \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$

$u = \tan^{-1}(x)$

limits: $u = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$u = \tan^{-1}(0)$

$= 0$

(1) finds upper / lower limits.

$\frac{du}{dx} = \frac{1}{1+x^2}$

$du = \frac{1}{1+x^2}\,dx$

$= \int_0^{\frac{\pi}{3}} \frac{1}{(1+x^2)} \times \frac{1}{(1+x^2)^{\frac{1}{2}}}\,du$

$= \int_0^{\frac{\pi}{3}} \frac{1}{(1+x^2)^{\frac{1}{2}}}\,du$

$x = \tan(u)$

$= \int_0^{\frac{\pi}{3}} \frac{1}{(1+\tan^2 u)^{\frac{1}{2}}}\,du$

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$= \int_0^{\frac{\pi}{3}} \frac{1}{(\sec^2 u)^{\frac{1}{2}}}\,du$

$= \int_0^{\frac{\pi}{3}} \frac{1}{\sec u}\,du \leftarrow$ (1) Correct integrand

$= \int_0^{\frac{\pi}{3}} \cos u\,du$

$= [\sin u]_0^{\frac{\pi}{3}}$

$= \sin(\frac{\pi}{3}) - \sin(0)$

$= \frac{\sqrt{3}}{2} \leftarrow$ (1) ✓ correct answer.

e) $V = \pi\int x^2\,dy$

$y = \ln(x-1)$

$e^y = x-1$

$x = e^y + 1$

when $x = k$

$y = \ln(k-1)$

$V = \pi\int_0^{\ln(k-1)} (e^y+1)^2\,dy$ (1)

$= \pi\int_0^{\ln(k-1)} e^{2y} + 2e^y + 1\,dy$

$= \pi\left[\frac{1}{2}e^{2y} + 2e^y + y\right]_0^{\ln(k-1)}$ (1) Correct integration OR limits

$= \frac{1}{2}e^{2\ln(k-1)} + 2e^{\ln(k-1)} + \ln(k-1) - (\frac{1}{2}e^{2(0)} + 2e^0 + 0)$

$= \left[\frac{1}{2}e^{2y} + 2e^{y} + y\right]_0$ (1) correct

$= \frac{1}{2}e^{2\ln(k-1)} + 2e^{\ln(k-1)} + \ln(k-1) - \left(\frac{1}{2}e^{2(0)} + 2e^{0} + 0\right)$

$= \frac{1}{2}e^{\ln((k-1)^2)} + 2e^{\ln(k-1)} + \ln(k-1) - \frac{1}{2} - 2$

$= \pi\left[\frac{1}{2}(k-1)^2 + 2(k-1) + \ln(k-1) - \frac{5}{2}\right]$

$\frac{\pi}{2}\left[k^2 - 2k + 1 + 4k - 4 + 2\ln(k-1) - 5\right]$

$\frac{\pi}{2}\left[k^2 + 2k - 8 + \ln((k-1)^2)\right] = \frac{\pi}{2}(27 + \ln(16))$

$k^2 + 2k - 8 = 27$

$k^2 + 2k - 35 = 0$

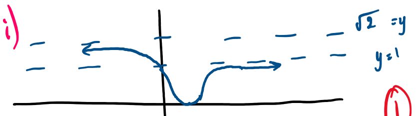
$(k+7)(k-5) = 0$

$k = 5$ or $k = -7$ but $\ln(k-1)$

$k - 1 > 0$

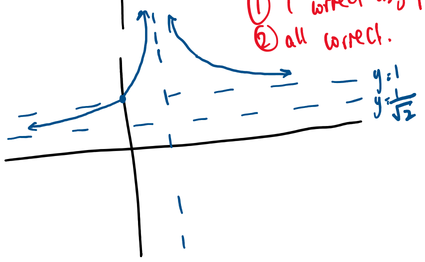
$\therefore k = 5.$ (1) Correct answer.

(14) a) i)



(1)

ii)



(1) 1 correct asymptote
(2) all correct.

b) $\frac{dV}{dt} = -h\sqrt{h}$

$V = Ah$

$V = 10h$

$\frac{dV}{dh} = 10$ (1)

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$ (1)

$= \frac{1}{10} \times -h^{\frac{3}{2}}$

$= \frac{-h^{\frac{3}{2}}}{10}$

$\frac{dt}{dh} = \frac{-10}{h^{\frac{3}{2}}}$

$\frac{dt}{dh} = -10h^{-\frac{3}{2}}$

$\int dt = \int -10h^{-\frac{3}{2}}\,dh$

$t = \frac{-10h^{-\frac{1}{2}}}{-\frac{1}{2}} + C$ (1)

$t = \frac{20}{\sqrt{h}} + C$

when $t = 0$, $h = 4$ (1)

$\frac{}{\sqrt{h}} + c$

when $t = 0$, $h = 4$ (1)

$0 = \frac{20}{\sqrt{4}} + c$

$c = -10$

$t = \frac{20}{\sqrt{h}} - 10$

Find when $V = 1L$

$1L = \frac{1}{1000} m^3$

$V = 10h$

$\frac{1}{1000} = 10h$

$h = \frac{1}{10000}$

$t = \frac{20}{\sqrt{\frac{1}{10000}}} - 10$

$= \frac{20}{\frac{1}{100}} - 10$

$= 2000 - 10$

$= 1990$ hours. (1)

c) i) $\underset{\sim}{r}(t) = \begin{bmatrix} 10\sqrt{2}t \\ 10\sqrt{2}t - 5t^2 + 1 \end{bmatrix}$

$x = 10\sqrt{2}t$

$10 = 10\sqrt{2}t$

$t = \frac{10}{10\sqrt{2}}$

$t = \frac{1}{\sqrt{2}}$ seconds the ball hits the wall.

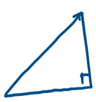
$y = 10\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) - 5\left(\frac{1}{\sqrt{2}}\right)^2 + 1$

$= 10 - \frac{5}{2} + 1$

$= 8.5m$ (1) ✓

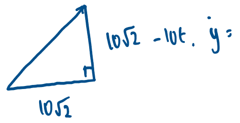
ii) $\underset{\sim}{v}(t) = \begin{bmatrix} 10\sqrt{2} \\ 10\sqrt{2} - 10t \end{bmatrix}$ (1) ✓

iii)



$10\sqrt{2} - 10t$. $\dot{y} =$

iii)



at $t = \frac{1}{\sqrt{2}}$

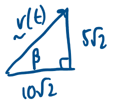
$10\sqrt{2} - 10\left(\frac{1}{\sqrt{2}}\right)$

$\dot{y} = 10\sqrt{2} - \frac{10}{\sqrt{2}}$

$\dot{y} = 10\sqrt{2} - \frac{10\sqrt{2}}{2}$

$\dot{y} = 10\sqrt{2} - 5\sqrt{2}$

$= 5\sqrt{2}$



$\tan\beta = \frac{5\sqrt{2}}{10\sqrt{2}}$

$\tan\beta = \frac{1}{2}$

$\beta = 26.565°$ (1) ✓

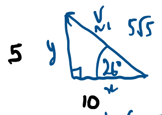
$|\underset{\sim}{u}| = \sqrt{(5\sqrt{2})^2 + (10\sqrt{2})^2}$

The b $= \sqrt{50 + 200}$

$= \sqrt{250}$

$= 5\sqrt{10}$ (1)

iv)

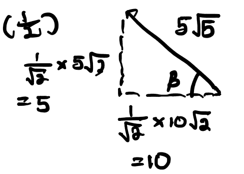


$\cos 26° = \frac{x}{5\sqrt{5}}$

$\underset{\sim}{v}_1 = \frac{1}{\sqrt{2}}(5\sqrt{10})$ ← conditions at moment of rebound — M1

$= 5\sqrt{5}$

$\beta = \tan^{-1}(\frac{1}{2})$



$\underset{\sim}{a}(t) = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$ ← integrates from $\underset{\sim}{a}(t)$ to find $\underset{\sim}{v}(w)$

$\underset{\sim}{v}_1(t) = \begin{bmatrix} c_1 \\ \cdot 10t + c_2 \end{bmatrix}$

$\therefore \underset{\sim}{v}_1(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \end{bmatrix}$

Let $t=0$ be time from ball hitting wall.

given $\underset{\sim}{r}(0) = \begin{bmatrix} 10 \\ 8.5 \end{bmatrix}$

at $t=0$

$c_1 = -5\sqrt{5}\cos(26°)$

$c_2 = 5\sqrt{5}\sin(26°)$

$\underset{\sim}{a}(t) = \begin{bmatrix} 0 \\ -10 \end{bmatrix}$

$\underset{\sim}{v}_1(t) = \begin{bmatrix} -5\sqrt{5}\cos(26°) \\ 10t + 5\sqrt{5}\sin(26°) \end{bmatrix}$

$\underset{\sim}{v}(t) = \begin{bmatrix} c_1 \\ -10t + c_2 \end{bmatrix}$

$\underset{\sim}{r}_1(t) = \begin{bmatrix} -5\sqrt{5}t\cos(26°) + c_3 \\ 5t^2 + 5\sqrt{5}t\sin(26) + c_4 \end{bmatrix}$

at $t=0$

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 5 \end{bmatrix}$

at $t=0$ $c_3 = 10$ (given)

$c_4 = 8.5$ (part i)

$\underset{\sim}{v}(t) = \begin{bmatrix} -10 \\ -10t + 5 \end{bmatrix}$ ← M1 integrates to get $\underset{\sim}{r}(t)$ expression.

$\underset{\sim}{r}_1(t) = \begin{bmatrix} -5\sqrt{5}t\cos(26°) + 10 \\ 5t^2 + 5\sqrt{5}t\sin(26°) + 8.5 \end{bmatrix}$

$\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\, dt$

$$= \int \begin{bmatrix} -10 \\ -10t + 5 \end{bmatrix} dt$$

$$= \begin{bmatrix} -10t + C_3 \\ -\frac{10t^2}{2} + 5t + C_4 \end{bmatrix}$$

$$= \begin{bmatrix} -10t + C_3 \\ -5t^2 + 5t + C_4 \end{bmatrix}$$

at $t = 0$

$C_3 = 10$

$C_4 = 8.5$

$$\underset{\sim}{r}(t) = \begin{bmatrix} -10t + 10 \\ -5t^2 + 5t + 8.5 \end{bmatrix}$$

when ball hits Q

$-10t + 10 = 0$

$t = 1$

$\underset{\sim}{r}(1) = \begin{bmatrix} 0 \\ 8.5 \end{bmatrix}$ as required ← M1 correct solution showing same y coord.